TIPTEKNO'18
TIP TEKNOLOJİLERİ KONGRESİ
8-10 Kasım 2018 / Gazi Magosa / KKTC

Chaotic Dynamics of a Jerk Function with Hyperbolic Tangent Nonlinearity

Hiperbolik Tanjant Doğrusalsızlığı İçeren Bir Jerk Fonksiyonunun Kaotik Dinamiği

Sibel ÖZBAL
Biyomedikal Mühendisliği
Yeditepe Üniversitesi
İstanbul, Türkiye
sibel.ozbal@std.yeditepe.edu.tr

Hayrettin Can SÜDOR
Biyomedikal Mühendisliği
Yeditepe Üniversitesi
İstanbul, Türkiye
sudorcan@gmail.com

Ali Ümit KESKİN
Biyomedikal Mühendisliği
Yeditepe Üniversitesi
İstanbul, Türkiye
auk@yeditepe.edu.tr

***Abstract*—This paper analyzes and reports the dynamical behavior of a jerk system used in biomedical modelling applications. This system employs a hyperbolic tangent function as a single source of nonlinearity. The system trajectories, Poincaré maps and Lyapunov exponents and spectrum, as well as Bifurcation diagram are presented to verify the chaotic dissipative behavior of the system, generating a double-scroll chaotic attractor.**

Keywords—Chaos, jerk function, Lyapunov exponents, bifurcation, Poincare map.

***Özetçe*—Bu makale, biyomedikal modelleme uygulamalarında kullanılan jerk sisteminin dinamik davranışını analiz etmekte ve raporlamaktadır. Bu sistem, bir hiperbolik tanjant fonksiyonunu tek bir doğrusalsızlık kaynağı olarak kullanır. Sistem yörüngeleri, Poincaré haritaları ve Lyapunov üsleri ve spektrumu ile birlikte Bifurcation diyagramı sunularak sistemin kaotik disipatif davranış gösterdiği ve çift kayan kaotik bir çekici üretmekte olduğu gösterilmiştir.**

Anahtar Kelimeler — Kaos, jerk fonksiyonu, Lyapunov üstelleri, bifurkasyon, Poincare haritası.

I. INTRODUCTION

Exposure to changes in motion can have significant biomechanical effects on the human body. One does not feel velocity, but only the acceleration, brought about by the force exerted by an object on his or her body.

In physics terms, the jerk is the third derivative of the position with respect to time.

Jerk should always be considered when vibration occurs and particularly when this excitation induces multi-resonant modes of vibration. It should also be considered at all times when a transition occurs such as: start up and shutdown; take-off and landing; and accelerating and decelerating. Acceleration without jerk is just a static load, and therefore constant acceleration alone could never cause vibration. During an orthopedic surgery, a surgeon can damage the bone or the instrument if the cutting tool starts vibrating. This vibration happens because of jerk, as the rapid changes in acceleration of a cutting tool can lead to harming results.

The human tolerance to acceleration has been measured and is well understood. But the human tolerance to jerk is not well understood.

A Jerk system has the structure $x' = y$, $y' = z$, $z' = f(x, y, z)$, where x, y, z are the states, and prime indicates differentiation. Consider one type of jerk equation of the form

$$x''' + ax'' + bx' + cx - dkf(x) = 0 \quad (1)$$

where $f(x)$ being a nonlinear function and $(a, b, c, d, k) \in \mathbb{R}$, with x denoting displacement function.

For simplicity, let $a = b = c = d$. Eq. (1) can be reduced into the following nonlinear system;

$$\boldsymbol{\phi}' = \boldsymbol{A\phi} + \boldsymbol{B} \quad (2a)$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -a & -a \end{bmatrix}, \quad (2b)$$

$$\boldsymbol{\phi}' = [x \quad y \quad z]^T \quad (2c)$$

$$\boldsymbol{B} = [0 \quad 0 \quad akf(x)]^T \quad (2d)$$

where T represents transpose operation. Here first term in the right side of $(2a)$ is the linear part of the system and second term is the nonlinearity vector, $\boldsymbol{B}$.

In this paper, we first analyse the linear part of the system and then report its general performance when the nonlinearity is a hyperbolic tangent function.

978-1-5090-2386-8/16/$31.00 ©2018 IEEE

Eq.(2) is a relatively general equation. For example, Sprott's jerk function [1] with quadratic nonlinearity is expressed as

$$j(x'',\quad x',\quad x) = -az - x \mp y^2$$

This is an autonomous dissipative system where a is a bifurcation parameter leading to chaos (in a very narrow range, $2.0168 < a < 2.0577$), and no simpler case of a jerk equation can exist. Disadvantage of Sprott's jerk system is that the quadratic nonlinearity is hard to implement electronically, and the circuit is delicate since the chaos exists in a quite limited range of bifurcation parameter.

Azar-Vaidyanathan jerk system [2] has two quadratic nonlinearities,

$$j(x'',\quad x',\quad x) = -ax - y + ax^2 + by^2$$

where $a = 2, b = 0.01$.

Jerk system of [3] has both exponential and multiplicative nonlinearity,

$$j(x'',\quad x',\quad x) = 5 - ay - az - e^x + bxy$$

Another chaotic jerk system [4] is described with a discontinuous nonlinearity,

$$j(x'',\quad x',\quad x) = -0.6z - y - 1.2x + 2\text{sgn}(x)$$

The studies in [5], [6] report oscillators using hyperbolic tangent function .

Lü et.al. [7] and Pachero et al [8] report the performance of a jerk system based on saturated nonlinear functions. Reference work [9] also study jerk systems using signum function.

Jerk systems can be realized with a compact electrical circuit structure. A jerk circuit has four connections at the node x, where the derivative of x is determined by the amplitude and polarity of y, and the amplitude and polarity of x influence the derivative of z. A study of chaotic flows with incomplete information transmission from the node x based on the jerk structure (hypogenetic chaotic jerk flows) are studied in [10].

II. ANALYSIS

The hyperbolic function, as a continuously differentiable function, is easier to analyze chaos. It is monotonically increasing for $-\infty < x < \infty$ and bounded, $-1 < \tanh(x) < 1$, which is a typical characteristic function that can replace piecewise linear functions particularly used in circuit design.

Equilibrium point of the linear part of the jerk system is obtained by solving 3D system, $(y = 0\,, z = 0, -a(x + y + z) = 0)$ which yields the origin as the equilibrium point

As for the nonlinear system, equating (2) to zero yields three equilibrium points,

$$E_o = (0,0,0), \qquad E_{1,2} = (\pm k, 0,0).$$

The Jacobian Matrix of this system is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a \cdot k(tanh^2(x) - 1) & -a & -a \end{bmatrix}$$

Evaluation of Jacobian Matrix at E_o for $a = 0.7,\ k = 10$ yields

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6.3 & -0.7 & -0.7 \end{bmatrix}$$

From which eigenvalues are obtained by solving the characteristic equation $|\lambda \boldsymbol{I} - \boldsymbol{I}| = 0$, where $\boldsymbol{I}$ stands for the identity matrix,

$$\lambda_1 = 1.531, \qquad \lambda_{2,3} = -1.115 \mp 1.694i$$

Evaluating Jacobian Matrix at $E_{1,2}$ for the same parameter values of a, k yields

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.7 & -0.7 & -0.7 \end{bmatrix}$$

From which eigenvalues are found as

$$\lambda_1 = -0.848, \qquad \lambda_{2,3} = 0.07399 \mp 0.9055i$$

These results indicate that origin is a saddle with index one, while two other equilibrium points are of spiral saddle of index two type, since real parts of complex eigenvalues have positive values. Only equilibria of saddle points $E_{1,2}$ are responsible for generating double scrolls in this jerk system, while E_0 connects the two chaotic scrolls. Figure 1 shows the linear part of jerk function, in the x-y phase plane, while Figure 2, Figure 3 and Figure 4 display simulated results for a double-scroll chaotic attractor of jerk system in different phase planes.

III. LYAPUNOV EXPONENTS (LES) FOR THE JERK FUNCTION

An approach to determine whether a system is chaotic or not is to compute its Lyapunov exponents. The sum of the Lyapunov exponents is the time-averaged divergence of the phase space velocity; hence any dissipative dynamical system will have at least one negative exponent, the sum of all of the exponents is negative. If a system has at least one positive Lyapunov exponent, and all the trajectories are ultimately bounded, then the system is chaotic [11]. The algorithm employed in this work for determining Lyapunov exponents is described in [12]. The three LEs computed are

$$L_1 = 0.1069,\ L_2 = -0.005,\ L_3 = -0.7520.$$

Hence, $L_1 > 0$ implies that the system is chaotic. The sum of these LEs is negative, which indicates a dissipative system. Since the sign of LEs are (+, 0, -), a strange attractor for the dissipative jerk system is under consideration. LE spectra is shown in Figure 5.

Bifurcation diagram of this function is displayed in Figure 6, which demonstrates a relatively large band for the bifurcation parameter of the system.

Poincare maps of the jerk function with hyperbolic tangent function as the nonlinearity term are shown in Figure 7.

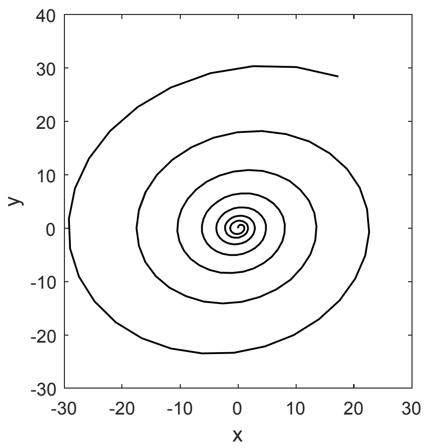

Figure. 1. Linear part of Jerk function, xy phase plane plot.

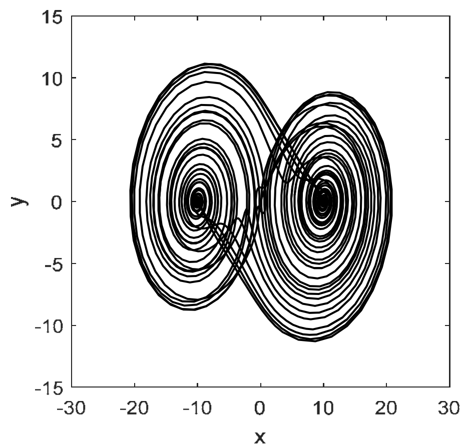

Figure. 2. Simulated results for a double-scroll chaotic attractor of jerk system with a = b = c = 0.7, k=10. The phase portrait projected on the x–y plane.

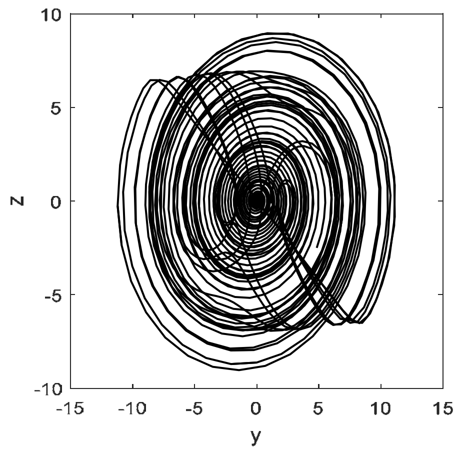

Figure. 3. Simulated results for a double-scroll chaotic attractor of jerk system with a = b = c = 0.7, k=10. The phase portrait projected on the z–y plane.

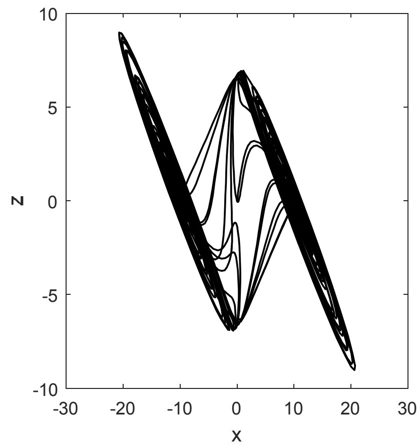

Figure. 4. Simulated results for a double-scroll chaotic attractor of jerk system with a = b = c = 0.7, k=10. The phase portrait projected on the x–z plane.

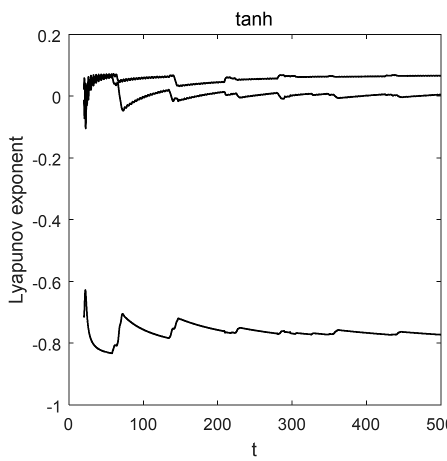

Figure. 5. LE=0.0615, 0.00, -0,7778, (Sum= - 0.7163), LDim=2.0937 a=0.7, k=10

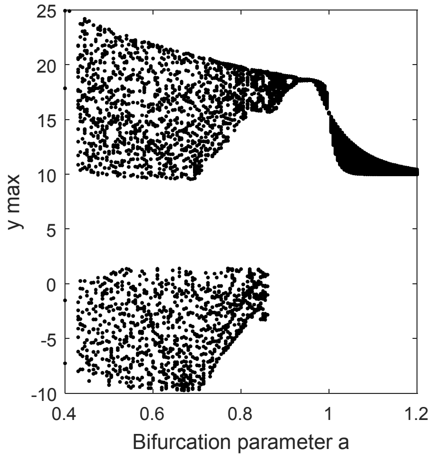

Figure. 6. Jerk system bifurcation diagram (k=10).

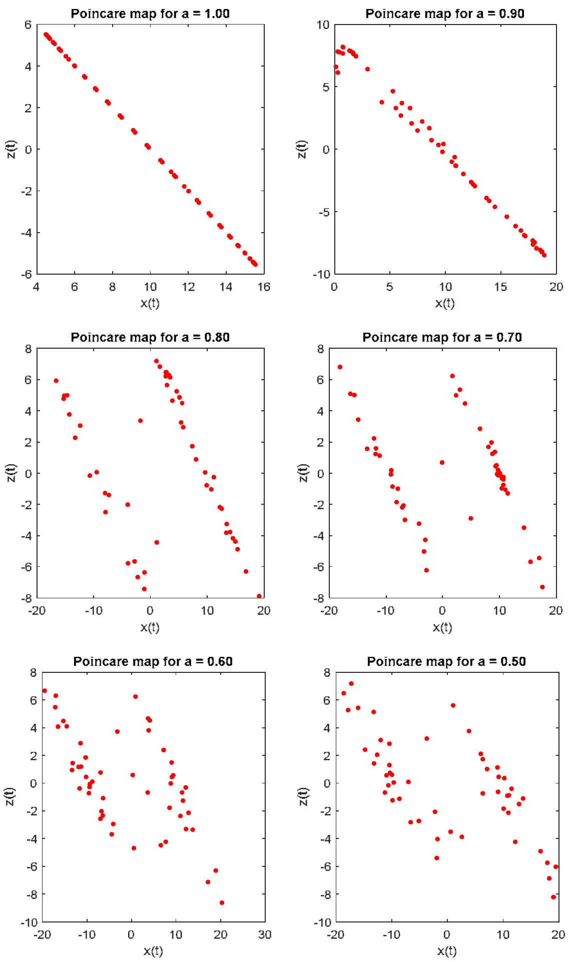

Figure. 7. Poincare maps of the jerk function with hyperbolic tangent function as the nonlinearity with different parameters, a.

IV. Conclusion

This paper has developed a nonlinear function approach for creating double-scroll chaotic attractors from a general jerk system. Dynamical behavior of such a jerk system which employs a hyperbolic tangent function as a single source of nonlinearity is described and examined. The system trajectories, Poincaré maps and Lyapunov exponents and spectrum, as well as bifurcation diagram are presented to verify the chaotic dissipative behavior of the system, generating a double-scroll chaotic attractor.

These results that are related to the particular jerk system presented in this paper can be used for impulsive or chaotic synchronization studies of such systems.

References

[1] JC Sprott, "Simplest dissipative chaotic flow" Physics Letters A, V228, No 4-5, pp 271-274,1997.

[2] Azar AT, Vaidyanathan S. (2016) Advances in Chaos Theory and Intelligent Control, Springer, pp:349-376.

[3] Vaidyanathan S, Azar AT (2016): Adaptive Backstepping Control and Synchronization of a Novel 3D Jerk System with an Exponential Nonlinearity. In Advances in Chaos Theory and Intelligent Control, Springer. pp: 249-274

[4] Bai E-W, Lonngren K E, Sprott JC, "On the synchronization of a class of Electronic circuits that exhibit Chaos", Chaos,Solitons and Fractals,13,1515-1521, 2002

[5] C. Wannaboon and T. Masayoshi, "An Autonomous Chaotic Oscillator Based on Hyperbolic Tangent Nonlinearity", 2015 15th International Symposium on Communications and Information Technologies (ISCIT), pp:323-326.

[6] F. Xu, P. Yu, "Chaos control and chaos synchronization for multi-scroll chaotic attractors generated using hyperbolic functions", J. Math. Anal. Appl. 362 (2010) 252–274.

[7] J.H. Lü, G.R. Chen, X.H. Yu, H. Leung, Design and analysis of multi-scroll chaotic attractors from saturated function series, IEEE Trans. Circuits Syst. I Regul. Pap. 51 (12) (2004) 2476–2490.

[8] J. M. Muñoz-Pacheco , D. K. Guevara-Flores, O. G. Félix-Beltrán, E. Tlelo-Cuautle ,J. E. Barradas-Guevara, and C. K. Volos, "Experimental Verification of Optimized Multiscroll Chaotic Oscillators Based on Irregular Saturated Functions", Complexity, V. 2018, Article ID 3151840.

[9] A. S. Elwakil and M. P. Kennedy, "Construction of classes of circuitindependent chaotic oscillators using passive-only nonlinear devices," IEEE Trans. Circuits Syst. I, vol. 48, pp. 289–307, Mar. 2001.

[10] C. Li, J. C. Sprott, H.Xing, "Hypogenetic chaotic jerk flows", Physics Letters A 380 1172–1177, 2016

[11] A.U. Keskin, "Ordinary Differential Equations for Engineers, Problems with MATLAB Solutions", Springer, p 678-679, 2019.

[12] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov Exponents from a Time Series," Physica D, Vol. 16, pp. 285-317, 1985.